So far, we've constructed sets from other sets as follows:

- as intersections or unions
- Taking the complement or the difference of two sets
- Taking the Cartesian product of sets
- Taking the power set of a set

We've used one more construction that we haven't yet discussed:

Axiom of choice: Let $\mathcal{C} = \{A_i\}_{i \in I}$ be a collection of nonempty sets (indexed by some set I). Then $\exists$ a set $S = \{x_i\}_{i \in I}$ s.t. $x_i \in A_i \;\; \forall \; i \in I$.

In other words, S is a set consisting of a single element from each set.

We implicitly used this in the proof that the countable union of countable sets is countable — do you see why?

Note that the axiom of choice is equivalent to the Cartesian product $\prod_{i \in I} A_i$ being nonempty (an element is a choice of an element from each A_i).

The axiom of choice seems innocent enough, but it was actually controversial originally. It proves crazy thing like the well-ordering theorem.

Def: An ordered set A is <u>well-ordered</u> if every nonempty subset of A has a smallest element. (e.g. $\mathbb{Z}_+$ w/ standard ordering)

Well-ordering theorem: If A is a set, $\exists$ an order relation on A that is a well-ordering.

Ex: Note that $\mathbb{Z}$ is not well-ordered under the standard ordering. However, we can define $<'$ as follows:

$$a <' b \iff \big(|a| < |b|\big) \text{ or } \big(|a| = |b| \text{ and } a < b.\big)$$

Then $0 <' -1 <' 1 <' -2 <' \dots$

Ex: $\mathbb{R}$ is certainly not well-ordered w/ the usual ordering. However, the well-ordering thm implies that such an ordering exists!